Relativity

e-content for B.Sc Physics (Hons)

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Question 1

Two spaceships pass each other, travelling in opposite directions. The speed of ship B as measured by a passenger in ship A is 0.96c. This passenger has measured the length of ship A as 100m and determines ship B is 30m. What are the lengths of two ships as measured by a passenger in ship B?

Answer

For time synchronized in the S' frame $L' = \frac{L}{\gamma}$ (this can be checked by using the Lorentz transformation for x) and for time synchronized in the S frame $L' = \gamma L$ where L and L' are the lengths measured in the S and S' frames.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{0.28} \tag{1}$$

When the passenger in ship B measures the length of A he must do so by measuring the two endpoints of A simultaneously according to his own clock. Therefore we transform the length measured by A using the equation $L' = \frac{L}{\gamma}$ which gives us $L'_A = 28$m.

The passenger in ship A calculated the length of B by measuring the endpoints of B simultaneously in his reference frame. Therefore the length measured by the passenger in B must be given by $L' = \gamma L$ which gives us $L'_B = 107.14$m.

Question 2

Two receivers, A and B, are positioned along the x axis. A source, moving at velocity βc along the x axis between A and B, radiates light of frequency f_0 in the rest frame of the source.

a) Show that to lowest order in β, the average frequency recorded by the two receivers is f_0.

Answer

Suppose that the light source is moving towards B and away from A. Then the Doppler shifted frequencies as measured at A and B are

$$f'_A = f_0\sqrt{\frac{1-\beta}{1+\beta}} \tag{2}$$

$$f'_B = f_0\sqrt{\frac{1+\beta}{1-\beta}} \tag{3}$$

We can use the following series expansions

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + ... \tag{4}$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + ... \tag{5}$$

So for f'_A we get

$$f'_A = (1-\beta)^{\frac{1}{2}}(1+\beta)^{-\frac{1}{2}}f_0 = f_0(1 - \beta + \frac{\beta^2}{2} + ...) \tag{6}$$

and for f'_B we get

$$f'_B = (1+\beta)^{\frac{1}{2}}(1-\beta)^{-\frac{1}{2}}f_0 = f_0(1 + \beta + \frac{\beta^2}{2} + ...) \tag{7}$$

To first order in β, $f'_A = (1-\beta)f_0$ and $f'_B = (1+\beta)f_0$. The average of these two frequencies is f_0.

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b) Show that there is a second order Doppler effect, i.e., a correction to f' that depends on β^2, and that this correction is the same when source and receiver approach and when they recede from each other.

Answer

We can see from equations (**??**) and (**??**) that the second order correction is the same in both f'_A and f'_B.

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c) Derive an expression for the average frequency recorded by the two receivers, including the second-order Doppler effect.

Answer

Taking the average of equations (**??**) and (**??**) we get $\frac{f'_A+f'_B}{2} = (1+\frac{\beta^2}{2})f_0$.

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d) Calculate $(f'_A + f'_B)/2f_0$ when $u = \beta c = 10^3 m/s$.

Answer

$\beta = \frac{u}{c} = \frac{10^3}{3(10^8)}$ so the β^2 term is negligible and $\frac{f'_A+f'_B}{2f_0} \approx 1$.

Question 3

A beam of radioactive particles emerges from the exit slit of an accelerator with a speed of 0.92c. Detectors placed 2.0 m and 4.0 m from the slit show that the intensity of the beam is $1.0 \times 10^8 particles/cm^2s$ at the first detector and $2.5 \times 10^7 particles/cm^2s$ at the second detector. a) Determine the half-life of the particles in their rest frame.

(Note: If there are N_0 radioactive particles at $t = 0$, the number that remain at time t is given by $N = N_0 e^{-t/\tau}$ and the half-life $T_{\frac{1}{2}}$ is the time it takes for the number of particles to decrease by 50 % i.e. $T_{\frac{1}{2}} = \tau \ln 2$.)

Answer

By definition, the half-life is the time it takes for the number of particles to halve. Since the intensity of the beam at the second detector is a quarter of that at the first detector, the time it takes the particles to travel from one detector to the other is twice the half-life in the laboratory frame.

$$T_{\frac{1}{2}} = \frac{t_2 - t_1}{2} \tag{8}$$

The half-life in the rest frame of the particles is $T'_{\frac{1}{2}} = \frac{T_{\frac{1}{2}}}{\gamma(v)}$. So,

$$T'_{\frac{1}{2}} = \frac{t_2 - t_1}{2\gamma(v)} = 1.42ns \tag{9}$$

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b) What is the beam intensity at the exit slit of the accelerator?

Answer

Since the distance between the exit slit and the first detector is the same as the distance between the two detectors it takes equally long to travel either distance. So,

$$N_2 = N_1 e^{-t'/\tau} \tag{10}$$

$$N_1 = N_0 e^{-t'/\tau} \tag{11}$$

$$\frac{N_1}{N_2} = \frac{N_0}{N_1} \tag{12}$$

$$N_0 = \frac{N_1^2}{N_2} = 4.0 \times 10^8 particles/cm^2 s \tag{13}$$

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c) Suppose that the accelerator is adjusted so that the particles emerge with a speed of 0.985c in the laboratory. If the beam intensity at the exit slit is the same as that calculated in part (b), what will the readings of the detectors be at 2.0 m and 4.0 m from the exit slit?

Answer

N_0 and $T'_{\frac{1}{2}}$ remain the same, but $\gamma(v)$ changes, so $T_{\frac{1}{2}} = \gamma(v)T'_{\frac{1}{2}}$ changes in value. Using the new value of the half-life in the laboratory frame in the equation

$$N = N_0 e^{-t \ln 2/T_{\frac{1}{2}}} \tag{14}$$

we can calculate

$$N_1 = 2.26 \times 10^8 particles/cm^2 s \tag{15}$$

$$N_2 = 1.28 \times 10^8 particles/cm^2 s \tag{16}$$

Question 4

A meterstick is positioned so that it makes an angle of 30 degrees with the x axis. Determine the length of the meterstick and its orientation as seen by an observor who is moving along the x axis with a speed of 0.8c.

Answer

The length along the x-axis is Lorentz contracted in the moving frame S'

$$L'_x = \frac{L_x}{\gamma} = \frac{L\cos 30}{\gamma} = 0.52m \tag{17}$$

The length in the y-direction, however, is unchanged

$$L'_y = L_y = L\sin 30 = 0.5m \tag{18}$$

Thus, the total length seen by the observor in the frame S' is

$$L' = \sqrt{{L'_x}^2 + {L'_y}^2} = 0.72m \tag{19}$$

and the angle θ between the meterstick and the x-axis is

$$\theta' = \arctan\frac{L'_y}{L'_x} = 43.9degrees \tag{20}$$

Question 5

A spaceship of proper length $L = 200m$ travels at a speed of 0.68c past a radio station. At the instant that the tail end of the ship passes the radio station as noted by an observor in the station, a signal is sent by the station's transmitter and subsequently detected by the receiver in the nose of the spaceship. Assume that the instant the nose of the spaceship passes the radio station, the clocks aboard the ship and at the station are synchronized to $t = t' = 0$. a) At what time, according to the clock aboard the spaceship, is the signal sent?

Answer

We define the frames S and S' to be the inertial frames respectively of the radio station and the spaceship. In this problem there are three events:

Event 1: the nose of the spaceship passes the radio station. At this time $t = t' = 0$. Furthermore, we can choose the origins of our x, x' axes at this point. (Note: from now on the nose of the ship is located at x' = 0)

Event 2: the tail of the spaceship reaches the point x = 0. At this instant the signal is emitted from the station.

Event 3: the signal reaches the nose of the ship (i.e. x' = 0)

To calculate the time t'_2 we use the Lorentz transform equation

$$t'_2 = \gamma(t_2 - vx_2/c^2) \tag{21}$$

The distance travelled by the ship in time t_2 is $\frac{L}{\gamma}$ at a speed of 0.68c. Therefore $t_2 = \frac{L}{v\gamma}$. (Note: L is the length of the ship in the S' frame)

$$t'_2 = \gamma(\frac{L}{v\gamma} - 0) = \frac{L}{v} = 0.98\mu s \tag{22}$$

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b) At what time, according to the clock at the radio station, is the signal received by the spaceship?

Answer

In the S frame, the signal reaches the spaceship at time t_3. By comparing the distance the signal and the ship travel (knowing that the ship got an early start) we get

$$c(t_3 - t_2) = v(t_3 - t_1) \tag{23}$$

$$t_3 = \frac{ct_2}{c - v} = 2.25\mu s \tag{24}$$

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c) At what time, according to the clock aboard the spaceship, is the signal received?

Answer

$$t'_3 = \gamma(t_3 - vx_3/c^2) \tag{25}$$

We know that x_3, the x-coordinate for event 3 is vt_3.

$$t'_3 = \gamma(t_3(1 - v^2/c^2)) = 1.65\mu s \tag{26}$$

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d) Where, according to an observor at the radio station, is the nose of the spaceship when the signal is received?

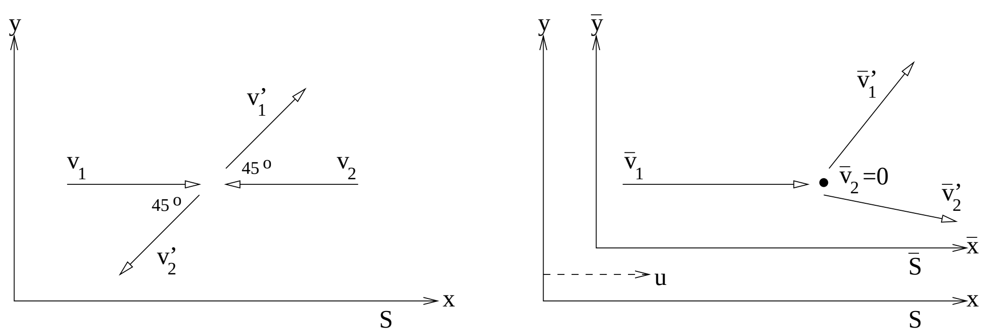
Answer

$$x_3 = \gamma(x'_3 + vt'_3) \tag{27}$$
$$x_3 = \gamma(vt'_3) = 458m \tag{28}$$

Question 6

Two particles of equal mass m collide head–on along the x–axis, each with a speed u. After the collision, the particles are observed to emerge at a $45°$ angle to the x–axis, each still having the same initial speed u. Using the relativistic expression for momentum, show that the total energy in the laboratory system, where one of the two particles is at rest before the collision, is the same before and after the collision.

Answer



The situation is as shown in the diagrams above. We have no prior knowledge of the velocities as measured in the lab frame, $\bar{S}$, so the right–hand diagram should be regarded as a schematic representation only. To calculate the momentum in the lab frame we must transform the initial (unprimed) and final (primed) velocities from the centre–of–mass frame, S, to the lab frame, and use the relativistic expression $\vec{p} = \gamma_v m_0 \vec{v}$. Several different γ's will appear in the problem; we use the following notation to distinguish them:

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The symbol m_0 represents the rest mass, that is, the mass as measured by an observer in the rest frame of each particle. We know that each particle has the same rest mass because we are told that they have the same mass in the centre–of–mass frame, where each has the same speed.

Then, in the centre–of–mass frame:

$$\begin{aligned} v_{1x} &= u, & v_{1y} &= 0, \\ v_{2x} &= -u, & v_{2y} &= 0, \\ v'_{1x} &= u/\sqrt{2}, & v'_{1y} &= u/\sqrt{2}, \\ v'_{2x} &= -u/\sqrt{2}, & v'_{2y} &= -u/\sqrt{2}, \end{aligned}$$

The velocities transform according to

$$\bar{v}_x = \frac{v_x + u}{1 + v_x u/c^2}, \qquad \bar{v}_y = \frac{v_y}{(1 + v_x u/c^2)\gamma_u},$$

so that, in the lab frame,

$$\begin{aligned} \bar{v}_{1x} &= \frac{2u}{1 + u^2/c^2}, & \bar{v}_{1y} &= 0, \\ \bar{v}_{2x} &= 0, & \bar{v}_{2y} &= 0, \\ \bar{v}'_{1x} &= u\frac{1 + 1/\sqrt{2}}{1 + u^2/(\sqrt{2}c^2)}, & \bar{v}'_{1y} &= \tfrac{u}{\sqrt{2}}\tfrac{\sqrt{1-u^2/c^2}}{1+u^2/(\sqrt{2}c^2)}, \\ \bar{v}'_{2x} &= u\frac{1 - 1/\sqrt{2}}{1 - u^2/(\sqrt{2}c^2)}, & \bar{v}'_{2y} &= -\tfrac{u}{\sqrt{2}}\tfrac{\sqrt{1-u^2/c^2}}{1-u^2/(\sqrt{2}c^2)}. \end{aligned}$$

Now all we have to do is compare the initial and final x– and y–components of the momentum. One way of doing so is to calculate the following quantities:

$$\begin{aligned} \Delta\bar{p}_x = (\bar{p}'_{1x} + \bar{p}'_{2x}) - (\bar{p}_{1x} + \bar{p}_{2x}) &= (m_0\gamma_{\bar{v}'_1}\bar{v}'_{1x} + m_0\gamma_{\bar{v}'_2}\bar{v}'_{2x}) - (m_0\gamma_{\bar{v}_1}\bar{v}_{1x} + m_0\gamma_{\bar{v}_2}\bar{v}_{2x}), \\ \Delta\bar{p}_y = (\bar{p}'_{1y} + \bar{p}'_{2y}) - (\bar{p}_{1y} + \bar{p}_{2y}) &= (m_0\gamma_{\bar{v}'_1}\bar{v}'_{1y} + m_0\gamma_{\bar{v}'_2}\bar{v}'_{2y}) - (m_0\gamma_{\bar{v}_1}\bar{v}_{1y} + m_0\gamma_{\bar{v}_2}\bar{v}_{2y}). \end{aligned}$$

We know everything in these expressions except the various γ's. They are easily calculated using the transformed velocities since, for instance,

$$\gamma_{\bar{v}_1} = \frac{1}{\sqrt{1-(\bar{v}_{1x}^2+\bar{v}_{1y}^2)/c^2}} = \frac{1+u^2/c^2}{1-u^2/c^2}.$$

The others are

$$\begin{aligned}\gamma_{\bar{v}_2} &= 1,\\ \gamma_{\bar{v}_1'} &= \frac{1+u^2/(\sqrt{2}c^2)}{1-u^2/c^2},\\ \gamma_{\bar{v}_2'} &= \frac{1-u^2/(\sqrt{2}c^2)}{1-u^2/c^2}.\end{aligned}$$

Substituting these into the expressions for $\Delta\bar{p}_x$ and $\Delta\bar{p}_y$ we find that

$$\begin{aligned}\Delta\bar{p}_x &= m_0\left[\frac{1+u^2/(\sqrt{2}c^2)}{1-u^2/c^2}\right]\left[\frac{u(1+1/\sqrt{2})}{1+u^2/(\sqrt{2}c^2)}\right]+m_0\left[\frac{1-u^2/(\sqrt{2}c^2)}{1-u^2/c^2}\right]\left[\frac{u(1-1/\sqrt{2})}{1-u^2/(\sqrt{2}c^2)}\right]\\ &\quad -m_0\left[\frac{1+u^2/c^2}{1-u^2/c^2}\right]\left[\frac{2u}{1+u^2/c^2}\right]\\ &= \frac{m_0u}{1-u^2/c^2}\left(1+\frac{1}{\sqrt{2}}+1-\frac{1}{\sqrt{2}}\right)-\frac{2m_0u}{1-u^2/c^2}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\Delta\bar{p}_y &= m_0\left[\frac{1+u^2/(\sqrt{2}c^2)}{1-u^2/c^2}\right]\left[\frac{u}{\sqrt{2}}\frac{\sqrt{1-u^2/c^2}}{1+u^2/(\sqrt{2}c^2)}\right]+m_0\left[\frac{1-u^2/(\sqrt{2}c^2)}{1-u^2/c^2}\right]\left[\frac{-u}{\sqrt{2}}\frac{\sqrt{1-u^2/c^2}}{1-u^2/(\sqrt{2}c^2)}\right]\\ &= 0.\end{aligned}$$

Hence, both the x– and y–components of momentum are conserved in the laboratory frame.