e-content for students

B. Sc.(honours) Part 1paper 2

Subject:Mathematics

Topic:Tangents & Normals(polar form)

RRS college mokama

# Tangents &Normals(polar form)

## Angle between Radius vector &Tangent

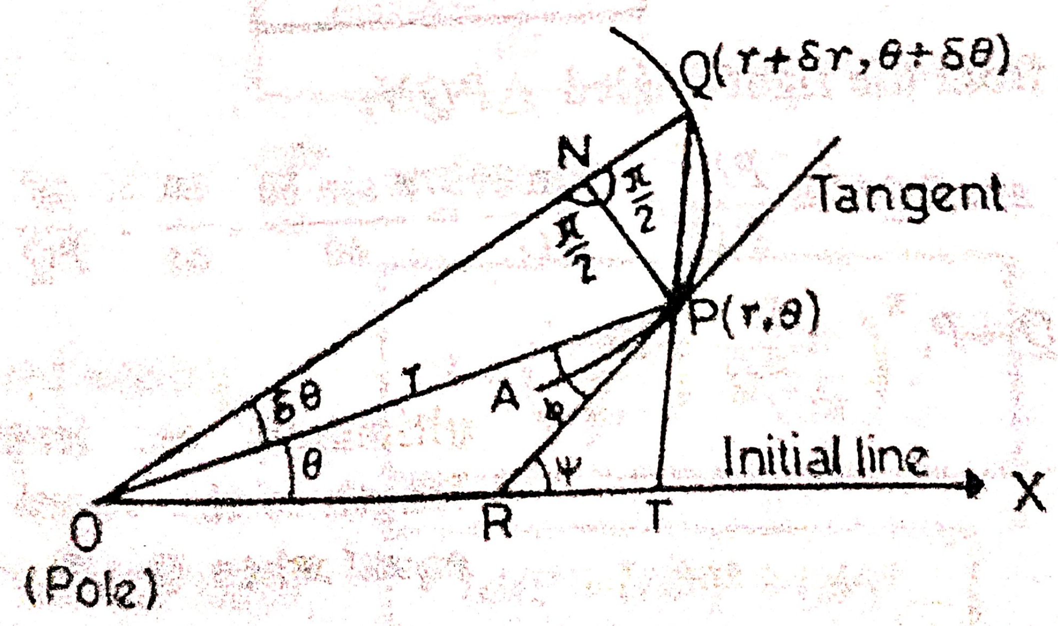
*To prove the formulae*

(i) $\tan\varphi = r\dfrac{d\theta}{dr}$; (ii) $\sin\varphi = \dfrac{r\,d\theta}{ds}$; (iii) $\cos\varphi = \dfrac{dr}{ds}$.

*where the symbols have their usual meanings.*

Let $O$ be the pole and $OX$ the initial line.

Let $A$ be the fixed point on the curve $r=f(\theta)$, from which the arc-length is measured and $P$ be any point $(r, \theta)$ on it such that arc $AP=s$. Let $Q$ be another point $(r+\delta r, \theta+\delta\theta)$ very close to $P$ so that arc $AQ=s+\delta s$. $\therefore$ arc $PQ=\delta s$.



Join $O$, $P$ and $O$, $Q$.

Draw $PN$ perpendicular from $P$ on $OQ$.

Draw tangent $PR$ to the curve at $P$. Join $P$ and $Q$.

$$OP = r,\ OQ = r + \delta r,\ \angle POX = \theta,\ \angle QOX = \theta + \delta\theta.$$

$$\therefore \quad \angle QOP = \delta\theta.$$

From right-angled $\triangle ONP$, $\sin \delta\theta = \dfrac{PN}{OP} = \dfrac{PN}{r}$.

$$\therefore \quad PN = r \sin \delta\theta.$$

Also $\cos \delta\theta = \dfrac{ON}{OP} = \dfrac{ON}{r}$. $\therefore \quad ON = r \cos \delta\theta.$

$$\therefore \quad QN = OQ - ON = r + \delta r - r\cos\delta\theta = r(1 - \cos\delta\theta) + \delta r$$

$$= 2r \sin^2 \frac{\delta\theta}{2} + \delta r. \qquad \text{.. (1)}$$

From right-angled $\triangle PNQ$,

$$\tan PQN = \frac{PN}{QN} = \frac{r \sin \delta\theta}{\delta r + 2r \sin^2 \dfrac{\delta\theta}{2}}$$

$$= \frac{r\left(\dfrac{\sin \delta\theta}{\delta\theta}\right)}{\dfrac{\delta r}{\delta\theta} + \dfrac{1}{2} r\,\delta\theta \left(\dfrac{\sin \dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}}\right)^2}.$$

Let $Q \to P$, so that $\delta\theta \to 0$ and $\angle PQN \to \angle RPO = \varphi$, where $\varphi$ is the angle between the radius vector $OP$ and the tangent $PR$.

Hence

$$\boxed{\tan \varphi = r \frac{d\theta}{dr}.}$$

Again from the right-angled $\triangle PQN$,

$$\sin PQN = \frac{PN}{PQ} = \frac{r \sin \delta\theta}{PQ} = \frac{r \sin \delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta s} \cdot \frac{\delta s}{PQ}. \qquad \text{.. (2)}$$

When $Q \to P$, $\dfrac{\delta s}{PQ} \to 1$.

($\because$ ultimately arc length $PQ$ and length of the chord $PQ$ become equal when $Q \to P$)

and $$\frac{\sin \delta\theta}{\delta\theta} \to 1 \quad \text{(by Calculus).}$$

$\therefore$ Taking limit, from (2), when $\delta\theta \to 0$,

$$\boxed{\sin \varphi = r\frac{d\theta}{ds}.}$$

Again from the right-angled $\triangle PQN$,

$$\cos PQN = \frac{QN}{PQ} = \frac{2r\sin^2\frac{\delta\theta}{2} + \delta r}{PQ}, \text{ from (1)}$$

$$= \frac{2r\sin^2\frac{\delta\theta}{2}}{\left(\frac{\delta\theta}{2}\right)^2} \cdot \left(\frac{\delta\theta}{2}\right)^2 \cdot \frac{1}{PQ} + \frac{\delta r}{PQ}$$

$$= \frac{1}{2}r \cdot \left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 \cdot \delta\theta \cdot \frac{\delta\theta}{PQ} + \frac{\delta r}{\delta s} \cdot \frac{\delta s}{PQ}$$

$$= \frac{1}{2}r\delta\theta \cdot \left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 \cdot \frac{\delta\theta}{\delta s} \cdot \frac{\delta s}{PQ} + \frac{\delta r}{\delta s} \cdot \frac{\delta s}{PQ}.$$

When $Q \to P$, $\frac{\delta s}{PQ} \to 1$ and $\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}} \to 1$, $\delta\theta \to 0$.

$\therefore$ Taking limit, when $\delta\theta \to 0$ we get

$$\boxed{\cos \varphi = \frac{dr}{ds}.}$$

# Polar sub tangent &polar sub normal



*To prove that the polar sub-tangent* $= r^2 \dfrac{d\theta}{dr}$

*and the polar sub-normal* $= \dfrac{dr}{d\theta}$.

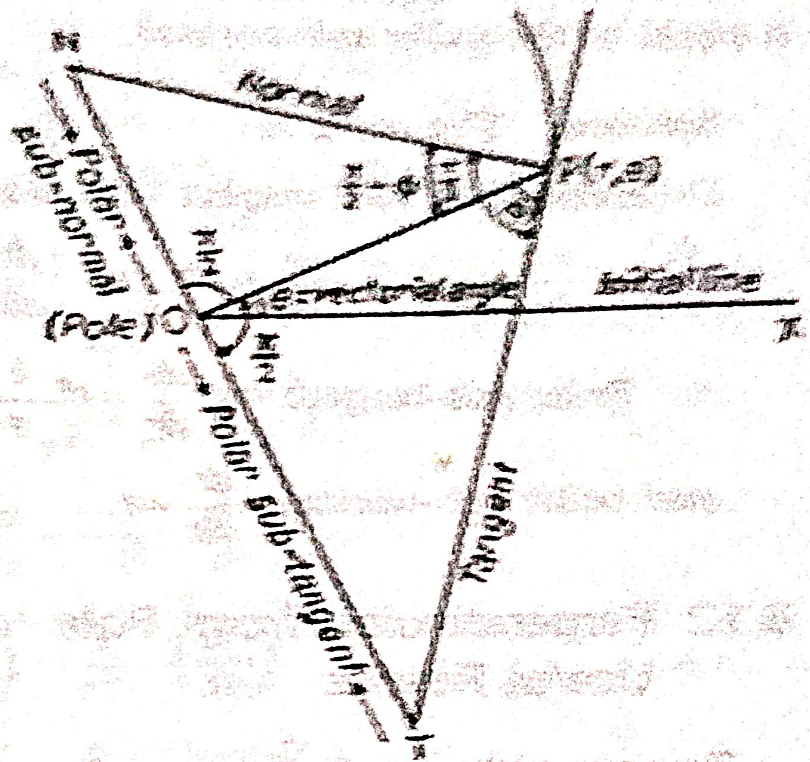
Let the equation of the curve be $r = f(\theta)$.

Let $O$ be the pole and $OX$ the initial line.

Let $PT$ and $PN$ be the tangent and the normal respectively to the curve at $P(r, \theta)$.

Join $O$ and $P$. Then $OP = r$ and $\angle POX = \theta$.

Through $O$ draw a straight line $ON$ perpendicular to the radius vector $OP$. Let this straight line meet the tangent at $T$ and the normal at $N$.




Then $OT$ is called the polar sub-tangent and $ON$ is called the polar sub-normal.

Now $\angle OPT = \varphi$, then $\angle OPN = \dfrac{\pi}{2} - \varphi$.

We know that $\tan\varphi = r\dfrac{d\theta}{dr}$.

From the right-angled $\triangle OPT$, $\tan\varphi = \dfrac{OT}{OP}$

or $r\dfrac{d\theta}{dr} = \dfrac{OT}{r}$; or $OT = r^2\dfrac{d\theta}{dr}$.

Hence, the polar sub-tangent $= r^2\dfrac{d\theta}{dr}$.

From the right-angled $\triangle PON$, $\tan\left(\dfrac{\pi}{2} - \varphi\right) = \dfrac{ON}{OP}$

i.e. $\cot\varphi = \dfrac{ON}{r}$; or $\dfrac{1}{r}\cdot\dfrac{dr}{d\theta} = \dfrac{ON}{r}$ or $ON = \dfrac{dr}{d\theta}$.

Hence the polar sub-normal $= \dfrac{dr}{d\theta}$.

Note If $u = \dfrac{1}{r}$, then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2} \cdot \dfrac{dr}{d\theta}$.

$\therefore$ Polar sub-tangent $= -\dfrac{d\theta}{du}$.

*Illustration.* Show that for the curve $r = e^{\theta}$, the polar sub-tangent is equal to the polar sub-normal.

*Solution.* The curve is $r = e^{\theta}$.

Differentiating with respect to $\theta$, we get

$$\frac{dr}{d\theta} = e^{\theta}; \quad \text{or} \quad \frac{dr}{d\theta} = r.$$

$\therefore$ Polar sub-tangent $= r^2 \dfrac{d\theta}{dr} = r^2 . \dfrac{1}{r} = r$,

and polar sub-normal $= \dfrac{dr}{d\theta} = r$.

# Perpendicular from pole onTangent

*To prove that*

(i) $p = r \sin \varphi$.

(ii) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$.

(iii) $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$ *where the symbols have their usual meanings.*

Let $ON$ be perpendicular drawn from the pole $O$ on the tangent at $P$.

Let $ON = p$.

$\angle OPQ = \varphi$,

$\angle OPN = \pi - \varphi$.

We know that
$$\tan \varphi = r \frac{d\theta}{dr}. \qquad \dots (1)$$

From the right-angled $\triangle OPN$, $\sin(\pi - \varphi) = \dfrac{ON}{OP}$

or
$$\sin \varphi = \frac{p}{r}; \quad \text{or} \quad p = r \sin \varphi.$$

or
$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2\varphi = \frac{1}{r^2}(1 + \cot^2\varphi)$$

or
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2} \cdot \frac{1}{\tan^2\varphi} = \frac{1}{r^2} + \frac{1}{r^2} \cdot \frac{1}{r^2} \cdot \left(\frac{dr}{d\theta}\right)^2.$$

Hence
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2.$$

(ii) Let $u = \dfrac{1}{r}$, then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2} \cdot \dfrac{dr}{d\theta}$;

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4} \cdot \left(\frac{dr}{d\theta}\right)^2.$$

Putting these values in $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4} \left(\dfrac{dr}{d\theta}\right)^2$, we get

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$