e-content for students

B. Sc.(honours) Part 1paper 1

Subject:Mathematics

Topic:Multiplication of two matrices

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Multiplication of two matrices

Def Let $A = [a_{ij}]$ be $m \times n$ matrix and let $B = [b_{jk}]$ be $n \times p$ matrix. Then the product of A and B denoted by AB is defined as the matrix $[c_{ik}]$ where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}$.

Obs. : In the above definition A is a $m \times n$ matrix and B is a $n \times p$ matrix and the product is a $m \times p$ matrix. This leads us to the following rule :

If A is a $m \times n$ matrix and B a $n \times p$ matrix, then AB will be a $m \times p$ matrix.

Ex. *Find the elements of the product $C = AB$,*

$$\text{where } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 3 & -2 \\ -4 & 3 \end{bmatrix}$$

Here A is a 3×4 matrix and B is a 4×2 matrix.

$\therefore$ The product AB will be a 3×2 matrix.

Now, c_{11} = sum of the products of the first row of A with the first column of B

$= 1 \times 1 + 2 \times (-2) + 3 \times 3 + 4 \times (-4)$

$= 1 - 4 + 9 - 16 = -10,$

c_{12} = sum of the products of the first row of A with the second column of B

$= 1 \times 2 + 2 \times 0 + 3 \times (-2) + 4 \times 3$

$= 2 + 0 - 6 + 12 = 8.$

Similarly, $c_{21} = 5 \times 1 + 4 \times (-2) + 3 \times 3 + 2 \times (-4)$

$= 5 - 8 + 9 - 8 = -2.$

$c_{22} = 5 \times 2 + 4 \times 0 + 3 \times (-2) + 2 \times 3$

$= 10 + 0 - 6 + 6 = 10$

$c_{31} = 0 \times 1 + 1 \times (-2) + 2 \times 3 + 3 \times (-4)$

$= -2 + 6 - 12 = -8$

$c_{32} = 0 \times 2 + 1 \times 0 + 2 \times (-2) + 3 \times 3$

$= 0 + 0 - 4 + 9 = 5.$

$$\text{Thus } C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} -10 & 8 \\ -2 & 10 \\ -8 & 5 \end{bmatrix}.$$

Algebraic laws for multiplication

Associative law . *If A and B are conformal for the product AB and B and C are conformal for the product BC, then*

$$(AB)C = A(BC).$$

Proof : Let A, B, C be the $m \times n$, $n \times p$ and $p \times q$ matrices and let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$.

Here A, B and C are conformal for the product AB and BC.

Now $(AB) = [a_{ij}] \times [b_{ij}] = \left[\sum_{k=1}^{n} a_{ik}\, b_{kj}\right]$ ...(I)

$= [\lambda_{ij}]$, say $i = 1, 2, 3, \ldots m$
$j = 1, 2, 3, \ldots p.$

We find that (AB) i.e., $[\lambda_{ij}]$ is a $m \times p$ matrix and since C is a $p \times q$ matrix; therefore (AB) and C are conformal for the product $(AB)C$ and $(AB)C$ is a $m \times q$ matrix.

Hence $(AB)C = [\lambda_{ij}] \times [c_{ij}]$

$$= \left[\sum_{l=1}^{p} \lambda_{il}\, c_{lj}\right] = \left[\sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik}\, b_{kl}\right) c_{lj}\right]; \text{ from (I)}$$

$$= \left[\sum_{l=1}^{p} \sum_{k=1}^{n} a_{ik}\, b_{kl}\, c_{lj}\right]; \quad i = 1, 2, 3, \ldots m; \quad j = 1, 2, 3, \ldots q.$$

Again $(BC) = [b_{ij}] \times [c_{ij}] = \left[\sum_{r=1}^{p} b_{ir}\, c_{rj}\right]$ ...(II)

$= [\delta_{ij}]$, say; $i = 1, 2, 3, \ldots n$; $j = 1, 2, 3, \ldots q$.

We find that (BC) i.e., $[\delta_{ij}]$ is a $n \times q$ matrix and since A is a $m \times n$ matrix, therefore A and (BC) are conformal for the product $A(BC)$ and $A(BC)$ is a $m \times q$ matrix.

Hence $A(BC) = [a_{ij}] \times [\delta_{ij}]$

$$= \left[\sum_{s=1}^{n} a_{is}\delta_{sj}\right] = \left[\sum_{s=1}^{n} a_{is} \left(\sum_{r=1}^{p} b_{sr}\, c_{rj}\right)\right]; \text{ from (II)}$$

$$= \left[\sum_{r=1}^{p} \sum_{s=1}^{n} a_{is}\, b_{sr}\, c_{rj}\right]; \quad i = 1, 2, 3, \ldots n; \quad j = 1, 2, 3, \ldots q.$$

Thus $(AB)C = A(BC)$.

We may write $(AB)C = A(BC) = ABC$.

Distributive law : *If A and B are conformal for the product AB, B and C are conformal for addition, then A(B + C) = AB + AC.*

Proof : Let A, B, C be the $m \times n$, $n \times p$ and $n \times p$ matrices and let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$.

Since B and C are conformal,

$\therefore \quad B + C = [b_{ij}] + [c_{ij}] = [b_{ij} + c_{ij}]$.

Now, $B + C$ is a $n \times p$ matrix. Therefore A and $B + C$ are conformal for the product $A(B + C)$.

Hence $A(B + C) = [a_{ij}] \times [b_{ij} + c_{ij}]$

$$= \left[\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})\right]; \; i = 1, 2, 3, \ldots m \quad j = 1, 2, 3, \ldots p$$

$$= \left[\sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}\right]$$

$$= \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right] + \left[\sum_{k=1}^{n} a_{ik} c_{kj}\right] \qquad \ldots(1)$$

But $AB = [a_{ij}] \times [b_{ij}] = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]; \; i = 1, 2, 3, \ldots m \quad j = 1, 2, 3, \ldots p$

and $AC = [a_{ij}] \times [c_{ij}] = \left[\sum_{k=1}^{n} a_{ik} c_{kj}\right]; \quad$ " $\quad$ " $\qquad \ldots(2)$

Therefore from (1) and (2), we have $A(B + C) = AB + AC$.

Similarly, $(B + C)D = BD + CD$, when D is a $p \times q$ matrix (say)

Cor. : $A(B - C) = AB - AC$.