e-content for students

B. Sc.(honours) Part 1paper 2

Subject:Mathematics

Topic:The Plane (3D geometry)

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General Equation of the First Degree

To prove that the general equation of the first degree (linear equation) in x, y, z represents a plane.

or, To prove that the equation

$$ax+by+cz+d=0,$$

where a, b, c, d are any given constants, a, b, c being not all zero, represents a plane.

Let the most general equation of the first degree in x, y, z be

$$ax+by+cz+d=0, \quad \quad (1)$$

where a, b, c, d are any given constants, a, b, c being not all zero.

We have to prove that the locus of this equation represents a plane.

Then, by definition of plane, our purpose is served provided we prove that every point on the line joining any two points on the locus also lies on the locus.

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two distinct points on the locus of (1).

Then $$ax_1+by_1+cz_1+d=0, \quad \quad (2)$$

and $$ax_2+by_2+cz_2+d=0. \quad \quad (3)$$

Multiplying (3) by k ($\neq -1$) and adding to (1), we get

$$a(x_1+kx_2)+b(y_1+ky_2)+c(z_1+kz_2)+d(1+k)=0.$$

Dividing by $1+k$, as $(1+k)\neq 0$, we get

$$a\left(\frac{x_1+kx_2}{1+k}\right)+b\left(\frac{y_1+ky_2}{1+k}\right)+c\left(\frac{z_1+kz_2}{1+k}\right)+d=0.$$

From this relation we find that the point

$$\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$$

also lies on the locus given by (1).

But for different values of k,

$$\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$$

gives different points on the line joining

$$A(x_1, y_1, z_1) \text{ and } B(x_2, y_2, z_2).$$

Therefore the line AB completely lies on the locus given by (1).

Hence, by definition of plane, the equation (1) represents a plane, provided a, b, c are not all zero.

Intercept form of Equation of a plane

To prove that the equation of the plane in the intercept form is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

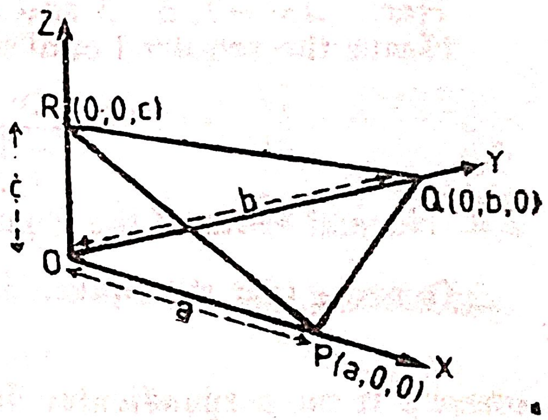
Let the equation of the plane be

$$Ax+By+Cz+D=0 \quad \ldots (1).$$

It cuts the x, y and z-axes at distances a, b and c respectively from the origin.

Then $OP=a$, $OQ=b$, $OR=c$.

Therefore, the co-ordinates of P, Q and R are respectively $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.



Since the plane (1) passes through the points $P(a, 0, 0)$, $Q(0, b, 0)$ and $R(0, 0, c)$, the co-ordinates of these points must satisfy the equation (1) of the plane.

$$\therefore \quad A.a+B.0+C.0+D=0; \quad \text{or} \quad A=-\frac{D}{a};$$

$$A.0+B.b+C.0+D=0; \quad \text{or} \quad B=-\frac{D}{b};$$

$$A.0+B.0+C.c+D=0; \quad \text{or} \quad C=-\frac{D}{c}.$$

Substituting the values of A, B and C in (1), we get

$$-\frac{D}{a}x-\frac{D}{b}y-\frac{D}{c}z+D=0$$

or

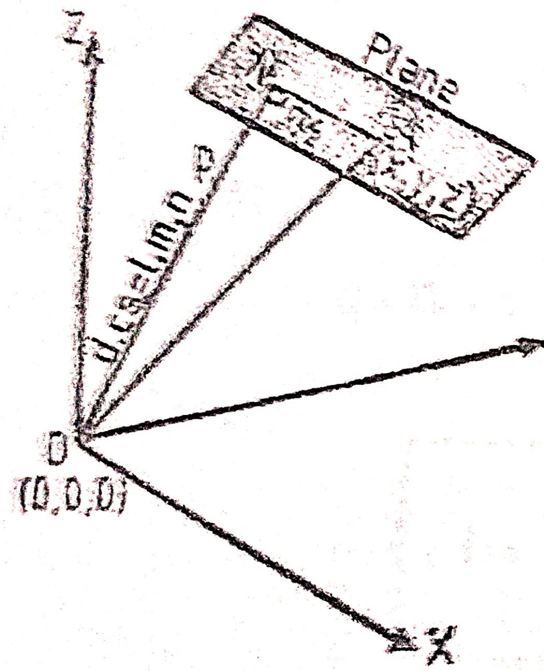
$$\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.}$$

This is the required equation of the plane.

Normal Form of the Equation of a p lane

Th:Obtain the equation of a plane i n normal form

From the origin $O(0, 0, 0)$ draw ON perpendicular upon the plane.



Let l, m, n be the direction cosines of this perpendicular ON and let $ON = p$.

Take any point $A(x, y, z)$ on the plane.

Now the points A and N are in the plane; therefore the line joining A and N must be in the plane. Again ON is normal to the plane; therefore ON must be perpendicular to NA.

$\therefore$ The projection of OA on $ON = ON$

or $$(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n=ON.$$

Here $x_1=0,\ y_1=0,\ z_1=0,\ x_2=x,\ y_2=y,\ z_2=z,\ ON=p.$

$$\therefore\quad (x-0)l+(y-0)m+(z-0)n=p$$

or

$$\boxed{lx+my+nz=p.}$$

This is the required equation of the plane as this relation holds for every point in the plane.

Reduction of general form of the equation of the plane

(i) To reduce the general equation

$$Ax+By+Cz+D=0$$

of the plane to the intercept form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

The general equation of the plane is

$$Ax+By+Cz+D=0; \quad \text{or} \quad Ax+By+Cz=-D.$$

Dividing by $-D$, we get

$$\frac{Ax}{-D}+\frac{By}{-D}+\frac{Cz}{-D}=\frac{-D}{-D}$$

or $$\frac{\frac{Ax}{A}}{\frac{-D}{A}}+\frac{\frac{By}{B}}{\frac{-D}{B}}+\frac{\frac{Cz}{C}}{\frac{-D}{C}}=1; \quad \text{or} \quad \frac{x}{\frac{-D}{A}}+\frac{y}{\frac{-D}{B}}+\frac{z}{\frac{-D}{C}}=1,$$

which is of the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

(ii) To reduce the general equation

$$Ax+By+Cz+D=0$$

of the plane to the normal form

$$lx+my+nz=p.$$

The general equation of the plane is

$$Ax+By+Cz+D=0 \quad \ldots (1).$$

The normal form of the equation of the plane is

$$lx+my+nz=p$$

or $$lx+my+nz-p=0 \quad \ldots (2),$$

where $l^2+m^2+n^2=1$ and p is positive.

If the equations (1) and (2) represent the same plane, then comparing their co-efficients, we get

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C}=\frac{-p}{D} \quad \ldots (3)$$

or l, m, n are proportional to A, B, C

i.e. the direction cosines of the normal to the plane $Ax+By+Cz+D=0$ are proportional to A, B, C.

Now from (3), $l=\dfrac{-pA}{D}$, $m=\dfrac{-pB}{D}$, $n=\dfrac{-pC}{D}$.

But $$l^2+m^2+n^2=1$$

or $\dfrac{p^2A^2}{D^2}+\dfrac{p^2B^2}{D^2}+\dfrac{p^2C^2}{D^2}=1$; or $\dfrac{p^2}{D^2}(A^2+B^2+C^2)=1$

or $p^2=\dfrac{D^2}{A^2+B^2+C^2}$; or $p=\pm\dfrac{D}{\sqrt{A^2+B^2+C^2}}$.

$$\therefore \quad l=\frac{\mp A}{\sqrt{A^2+B^2+C^2}}, \quad m=\frac{\mp B}{\sqrt{A^2+B^2+C^2}},$$

$$n=\frac{\mp C}{\sqrt{A^2+B^2+C^2}}.$$

Substituting these values in (2), we get the general equation

$$Ax+By+Cz+D=0$$

of the plane in the normal form as

$$\pm\frac{A}{\sqrt{A^2+B^2+C^2}}x\pm\frac{B}{\sqrt{A^2+B^2+C^2}}y\pm\frac{C}{\sqrt{A^2+B^2+C^2}}z$$

$$=\mp\frac{D}{\sqrt{A^2+B^2+C^2}} \quad \cdots (4),$$

the sign being so chosen that p, that is, $\mp\frac{D}{\sqrt{A^2+B^2+C^2}}$, is always positive.