e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 2

Subject:Mathematics

Topic:Reduction formulas of integration

Reduction formulas

Reduction Formulas for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$

In this sub-section we will consider integrands which are powers of either sinx or cosx. Let's take a power of sinx first. For evaluating $\int \sin^n x dx$, we write

$$I^n = \int \sin^n x dx = \int \sin^{n-1} x \sin x dx, \text{ if } n > 1.$$

Taking $\sin^{n-1}x$ as the first function and sin x as the second and integrating by parts, we get

$$I_n = -\sin^{n-1} x \cos x - (n-1) \int \sin^{n-2} x \cos x (-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) [\int \sin^{n-2} x (1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) [\int \sin^{n-2} x\, dx - \int \sin^n x)\, dx]$$

$$= -\sin^{n-1} x \cos x + (n-1) [I_{n-2} - I_n]$$

Hence,

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

That is, $nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$ Or

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the reduction formula for $\int \sin^n x dx$ (valid for $n \geq 2$).

Example We will now use the reduction formula for $\int \sin^n x dx$ to evaluate the definite integral, $\int_0^{\pi/2} \sin^5 x dx$. We first observe that

$$\int_0^{\pi/2} \sin^n x dx = \left. \frac{-\sin^{n-1} x \cos x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$= \frac{n-1}{n} \int^{\pi/2} \sin^{n-2} x dx, \; n \geq 2.$$

Thus,

$$\int_0^{\pi/2} \sin^5 x dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x dx$$

$$= \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x dx$$

$$= \frac{8}{15} \left. (-\cos x) \right]_0^{\pi/2}$$

$$= \frac{8}{15}$$

Let us now derive the reduction formula for $\int \cos^n x dx$. Again let us write

$$I_n = \int \cos^n x dx = \int \cos^{n-1} x \cos x dx, \; n > 1.$$

Integrating this integral by parts we get

$$I_n = \int \cos^{n-1} x \sin x - \int (n-1) \cos^{n-2} x \; (-\sin x). \sin x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1)(I_{n-2} - I_n)$$

By rearranging the terms we get

$$I_n = \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

This formula is valid for $n \geq 2$. What happens when $n = 0$ or 1? You will agree that the integral in each case is easy to evaluate.

As we have observed in Example 2.5

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx, \ n \geq 2.$$

Using this formula repreatedly we get

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4} \cdots\cdots \frac{4}{5}.\frac{2}{3} \int_0^{\pi/2} \sin x \, dx, \text{ if } n \text{ is an odd number, } n \geq 3. \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4} \cdots\cdots \frac{3}{4}.\frac{1}{2} \int_0^{\pi/2} dx, \text{ if } n \text{ is an even number, } n \geq 2. \end{cases}$$

This means

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2} \cdots\cdots \frac{4}{5}.\frac{2}{3} \text{ if } n \text{ is odd, and } n \geq 3 \\ \frac{n-1}{n}.\frac{n-3}{n-2} \cdots\cdots \frac{3}{4}.\frac{1}{2} \frac{\pi}{2}, \text{ if } n \text{ is even } n \geq 2 \end{cases}$$

We can reverse the order of the factors, and write this as

$$\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{2}{3} \frac{4}{5} \cdots\cdots \frac{n-3}{n-2} . \frac{n-1}{n}. \text{ if } n \text{ is odd, } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n} . \frac{\pi}{2}. \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

Arguing similarly for $\int_0^{\pi/2} \cos^n x dx$ we get

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{2}{3}.\frac{4}{5} \cdots\cdots \frac{n-1}{n}. \text{if } n \text{ is odd, and } n \geq 3 \\ \frac{1}{2} \frac{3}{4} \cdots\cdots \frac{n-1}{n}.\frac{\pi}{2} \text{ if } n \text{ is even, } n \geq 2 \end{cases}$$

Reduction Formulas for $\int \tan^n x dx$ and $\int \sec^n x dx$

In this sub-section we will take up two other trigonometric functions : tanx and secx. This is, we will derive the reduction formulas for $\int \tan^n x dx$ and $\int \sec^n x dx$. To derive a reduction formula for $\int \tan^n x dx$, $n > 2$. we start in a slightly different manner. Instead of writing $\tan^n x = \tan x \tan^{n-1} x$, as we did in the case of $\sin^n x$ and $\cos^n x$,. we shall write $\tan^n x = \tan^{n-2} x \tan^2 x$. You will shortly see the reason behind this. So we write

$$I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx.$$

$$= \int \tan^{n-2} x(\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \qquad \text{......(2)}$$

You must have observed that the second integral on the right hand side is I_{n-2}. Now in the first integral on the right hand side, the integrand is of the form $[f(x)]^m.f'(x)$

As we have seen in Unit 11,

$$\int [f(x)]^m f'(x)dx = \frac{[f(x)]^{m+1}}{m+1} + c$$

Thus, $\int \tan^{n-2} x \sec^2 x dx = \frac{\tan^{n-1} x}{n-1} + c$

Therefore, (2) give $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$.

Thus the reduction formula for $\int \tan^n x \, dx$ is

$$\int \tan^n x dx = I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

To derive the reduction formula for $\int \sec^n xdx \ (n > 2)$, we first write $\sec^n x = \sec^{n-2} x \sec^2 x$,

and then integrate by parts. Thus

$$I_n = \int \sec^n xdx = \int \sec^{n-2} x \sec^2 xdx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x, \sec x \tan^2 xdx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 xdx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \ (I_n - I_{n-2})$$

After rearranging the terms we get

$$\int \sec^n xdx = I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

These formulas for $\int \tan^n xdx$ and $\int \sec^n xdx$ are valid for $n > 2$. For $n = 0$, 1 and 2, the integrals $\int \tan^n xdx$ and $\int \sec^n xdx$ can be easily evaluated.

Exampet's ·(Let's calculate i) $\int_0^{\pi/4} \tan^5 x dx$ and ii) $\int_0^{\pi/4} \sec^6 x\, dx$

i) $\int_0^{\pi/4} \tan^5 x dx = \left.\frac{\tan^4 x}{4}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^3 x dx$

$= \frac{1}{4} - \left.\frac{\tan^2 x}{x}\right]_0^{\pi/4} + \int_0^{\pi/4} \tan x dx$

$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$

$= -\frac{1}{4} - \ln (\cos x)\Big]_0^{\pi/4}$

$= -\frac{1}{4} - \ln \frac{1}{\sqrt{2}} + \ln 1$

$= -\frac{1}{4} \ln \sqrt{2}$

ii) $\int_0^{\pi/4} \sec^6 x dx = \left.\frac{\sec^4 x \tan x}{5}\right]_0^{\pi/4} + \frac{4}{5} \int_0^{\pi/4} \sec^4 x dx$

$= \frac{4}{5} + \frac{4}{5} \left\{ \left.\frac{\sec^2 x \tan x}{3}\right]_0^{\pi/4} + \frac{2}{3} \int_0^{\pi/4} \sec^2 x dx \right\}$

$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15} \int_0^{\pi/4} \sec^2 x dx$

$= \frac{4}{3} + \frac{8}{15} \tan x \Big]_0^{\pi/4} = \frac{28}{15}$